Research Division
Federal Reserve Bank of St. Louis
Working Paper Series

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Working Paper 2015-039A
http://research.stlouisfed.org/wp/2015/2015-039.pdf

November 2015

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Why Are Exchange Rates So Smooth? A Segmented Asset Markets Explanation

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November 2015

Abstract

Empirical work on asset prices suggests that pricing kernels have to be almost perfectly correlated across countries. If they are not, real exchange rates are too smooth to be consistent with high Sharpe ratios in asset markets. However, the cross-country correlation of macro fundamentals is far from perfect. We reconcile these empirical facts in a two-country stochastic growth model with segmented markets. A large fraction of households either do not participate in the equity market or hold few equities, and these households drive down the cross-country correlation in aggregate consumption. Only a small fraction of households participate in international risk sharing by frequently trading domestic and foreign equities. These active traders are the marginal investors, who impute the almost perfect correlation in pricing kernels. In our calibrated economy, we show that this mechanism can quantitatively account for the excess smoothness of exchange rates in the presence of highly volatile stochastic discount factors.

JEL codes: G15, G12, F31, F10.

Keywords: asset pricing, market segmentation, exchange rate, international risk sharing.

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1 Introduction

A striking disconnect exists in international finance between the evidence gathered from asset prices and that gathered from quantities. When markets are complete, no arbitrage implies that the percentage rate of depreciation of the real exchange rate (RER) is given by the difference between the domestic and the foreign stochastic discount factors. We know from the data on RERs and asset prices that the volatility of RERs is much smaller than the volatility of the stochastic discount factors. Hence, the evidence from asset markets implies that the stochastic discount factors are highly correlated across countries. In fact, Brandt, Cochrane, and Santa-Clara (2006) conclude that the correlation of the stochastic discount factors is close to one. However, the quantity data paint a different picture. In standard representative agent models, the correlation of pricing kernels is identical to the correlation of aggregate consumption growth. Empirically, the correlation of aggregate consumption growth is below 50% for most industrialized country pairs. In this paper, we address this disconnect between prices and quantities in international finance.[1]

Household finance may hold the key to this disconnect. Standard macro-finance models assume that aggregate risk has been distributed efficiently across households, but in our model most households do not bear their share of aggregate risk. In equilibrium, a large amount of aggregate risk is borne by a relatively small number of sophisticated investors, who participate in both domestic and foreign equity markets. Thus, country-specific risk is concentrated among a small pool of sophisticated domestic and foreign investors. These investors achieve a higher degree of risk sharing among themselves than the average investor in these countries. Hence, the marginal investor's consumption growth is highly correlated across countries, but the average investor's is not. This mechanism can quantitatively account for the excess smoothness of the RER in the presence of stochastic discount factors that satisfy the Hansen-Jagannathan bounds.

Our approach is firmly grounded in the empirical evidence on household finance. These studies have found that most households do not purchase all assets available on the menu.[2] In fact, the composition of household asset holdings varies greatly across households even in a developed country like the United States. Only 50% of U.S. households participate in the equity market, according to the 2010 Survey of Consumer Finance. The nonparticipation rate is even higher in other developed countries. Even among the participants in the equity market, many investors trade

[1]When markets are incomplete, the percentage rate of depreciation of the RER is not identical to the difference between the domestic and the foreign stochastic discount factors. However, Lustig and Verdelhan (2015) show that market incompleteness cannot quantitatively resolve these puzzles without largely eliminating currency risk premia.

[2]See Guiso and Sodini (2012) for an excellent survey of this literature.

only infrequently and do not rebalance their portfolios. The heterogeneity in observed portfolio choices is consistent with a highly uneven distribution of risk across investors. In particular, the investors who do not participate in the equity market bear no aggregate risk.

In the quantitative exercise, we parameterize our model to match the world economy, in which the United States is the home country and the foreign country is the sum of France, Germany, Japan, and the United Kingdom. In our benchmark economy, we find that the international correlation of the pricing kernels exceeds 97%, while the international correlation of consumption growth is only 17%. Despite the high volatility of the pricing kernels, the high correlation of pricing kernels together with a high market price of risk yields a 9.4% standard deviation of RER depreciation, which is close to the data.

International trade in goods markets plays an essential role in solving this puzzle. Without goods trade, domestic households would not receive payments from abroad and hence would choose not to hold foreign assets. The importance of goods trade for international risk sharing is well documented by Fitzgerald (2012). She finds that frictions in the goods markets limit risk sharing for both developed and developing countries. These frictions in the goods markets generate a home bias in consumption. However, for analytical simplicity, we do not introduce trade friction parameters. Instead, we model a home bias in consumption by the preferences specification and the introduction of the nontraded good. In our quantitative exercise, we illustrate that a home bias in consumption is necessary for generating volatile exchange rates.

Some work related to this puzzle that modifies the preferences or the properties of cash flow growth in a standard representative agent model. Colacito and Croce (2011) endow the representative investor with non-time-separable preferences that impute a concern about long-run risk in consumption. When long-run risk is highly correlated across countries, the pricing kernels become highly correlated even though aggregate consumption growth is not. Farhi and Gabaix (2008) rely on correlated disaster risk instead. In related work, Stathopoulos (2012) analyzes a model in which the representative investor has preferences with external habit persistence. External habit formation creates an accumulative effect on pricing kernels from the past history of consumption growth that also generates highly correlated pricing kernels, despite the low correlation in current consumption growth.

Our main contribution to the literature is the integration of the micro evidence on household portfolio choices into a general equilibrium model to solve the exchange rate smoothness puzzle first articulated by Brandt, Cochrane, and Santa-Clara (2006). Our model does not require nonstandard preferences or aggregate risk specifications. Instead, the mechanism in our model relies on the skewness of the cross-sectional distribution of aggregate risk. To be precise, aggregate risk is concentrated among a small fraction of households that also share aggregate risk with a small fraction of foreign households. The risk distribution is driven by heterogeneous portfolio choices, which are strongly supported by the empirical evidence.

In addition, our study contributes to the emerging literature that integrates international portfolio choice into international macroeconomics. Specifically, we demonstrate the importance of household portfolio heterogeneity in open economies, whereas the majority of open-economy macroeconomic models rely on a representative agent framework. Recent studies by Coeurdacier and Rey (2013) and Pavlova and Rigobon (2010) are prominent examples of this line of work. Most international macroeconomic models assume either incomplete markets with only one asset or a complete market environment without heterogeneity in trading technologies. Although a complete menu of assets is traded in our model, we allow for heterogeneity in household trading technologies because it is strongly supported by the data. Our model creates a role for the heterogeneity in trading technologies that we observe in the data. Therefore, our model provides a laboratory for studying the consequences of the cross-sectional heterogeneity in portfolio choices on exchange rates.

The rest of our study is organized as follows. The next section describes our model. The quantitative results and counterfactual exercises are detailed in Section 3. Section 4 concludes our study.

2 The Model

2.1 Environment

We consider an endowment economy with two countries, home and foreign. There are a large number of agents in each country. Each country is endowed with a nontraded good and an export good. For simplicity, we assume that home households consume the nontraded good and the foreign export good. Likewise, foreign households consume the nontraded good and the home export good.

Time is discrete, infinite, and indexed by $t \in [0, 1, 2, ...)$. To have a stationary economy, we assume an identical average growth rate for each country, while the actual growth rate may deviate from the average one. More specifically, let $\ln m_t$ and $\ln m_t^*$ be the percentage deviation of endowment from the growth trend. Then, the country-specific endowment in period t is

$$\begin{aligned}\ln Y_t &= t\ln\overline{g} + \ln m_t, \\ \ln Y_t^* &= t\ln\overline{g} + \ln m_t^*,\end{aligned}$$

where $\overline{g}$ is the average growth rate of endowment of both countries. The output growth process is therefore governed by the evolution of m, which follows the following AR(1) process:

$$\begin{aligned}\ln m_{t+1} &= \rho\ln m_t + \varepsilon_{t+1},\ \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2) \\ \ln m_{t+1}^* &= \rho\ln m_t^* + \varepsilon_{t+1}^*,\ \varepsilon_{t+1}^* \sim N(0, \sigma_\varepsilon^{*2}).\end{aligned}$$

Let z^t denote the history of aggregate states up to period t. In each country, a constant fraction λ of the endowment is the nontraded good, and the rest is the export good:

$$\begin{aligned}Y_n(z^t) &= \lambda Y(z^t) \text{ and } Y_x(z^t) = (1-\lambda)Y(z^t) \\ Y_n^*(z^t) &= \lambda Y^*(z^t) \text{ and } Y_x^*(z^t) = (1-\lambda)Y^*(z^t),\end{aligned}$$

where Y_n, Y_x, Y_n^*, and Y_x^* denote endowments of home nontraded, home export, foreign nontraded, and foreign export goods, respectively.

Aggregate income is divided into two parts: diversifiable income and nondiversifiable income. Claims to the diversifiable income can be traded in financial markets while claims to nondiversifiable income cannot. We assume a constant share of nondiversifiable income, α, across countries and time. The nondiversifiable component is subject to idiosyncratic stochastic shocks. These shocks are i.i.d. across households in a country. We use η_t and η_t^* to denote the idiosyncratic shock in period t for the home and the foreign countries respectively. Then, η^t and $\eta^{*,t}$ denote the history of idiosyncratic shocks of home and foreign households, respectively.

We use $\pi(z^t, \eta^t)$ to denote the unconditional probability of that state (z^t, η^t) will be realized. The events are first-order Markov and their probabilities are assumed to be independent between

z shocks and η or η^* shocks:

$$\begin{aligned}\pi(z^{t+1}, \eta^{t+1}|z^t, \eta^t) &= \pi(z_{t+1}|z^t)\pi(\eta_{t+1}|\eta_t)\\ \pi(z^{t+1}, \eta^{*,t+1}|z^t, \eta^{*,t}) &= \pi(z_{t+1}|z^t)\pi(\eta^*_{t+1}|\eta^*_t).\end{aligned}$$

2.2 Preferences

The household derives utility from consuming composites of goods,

$$\sum_{t=1}^{\infty}\beta^t \sum_{(z^t,\eta^t)} \frac{c(z^t, \eta^t)^{1-\gamma}}{1-\gamma}\pi(z^t, \eta^t), \quad (1)$$

where $\gamma > 0$, $0 < \beta < 1$. The parameter γ denotes the coefficient of relative risk aversion, β is the time discount factor, and $c(z^t, \eta^t)$ is the consumption basket. The home consumption basket is a Cobb-Douglas composite of the nontraded good c_n and the foreign export good c^*_x,

$$c(z^t, \eta^t) = c_n(z^t, \eta^t)^{\theta} c^*_x(z^t, \eta^t)^{1-\theta}.$$

The parameter $\theta \in [0, 1]$ represents a home bias in consumption and governs the relative preferences over the nontraded and foreign export goods.

Similarly, the preferences for the foreign households are given by

$$U\left(\{c^*\}\right) = \sum_{t=1}^{\infty}\beta^t \sum_{(z^t,\eta^t)} \frac{c^*(z^t, \eta^t)^{1-\gamma}}{1-\gamma}\pi(z^t, \eta^t), \quad (2)$$

where $\gamma > 0$, $0 < \beta < 1$. The foreign basket is similarly defined as

$$c^*(z^t, \eta^t) = c^*_n(z^t, \eta^t)^{\theta} c_x(z^t, \eta^t)^{1-\theta}.$$

2.3 Correlation of Consumption Growth

Given the preferences, the aggregate consumption basket becomes a composite of the home non-traded good and the foreign export good endowment:

$$C(z^t) = Y_n(z^t)^{\theta} Y_x^*(z^t)^{1-\theta}. \tag{3}$$

Likewise, foreign aggregate consumption is a composite of the foreign nontraded good and the home export good:

$$C^*(z^t) = Y_n^*(z^t)^{\theta} Y_x(z^t)^{1-\theta}. \tag{4}$$

Although trade flows are exogenously determined by the preferences specification, the RER and consumption growth correlation are endogenous to shocks on endowment. To see why, notice that the resource constraints in (3) and (4) and the endowment imply that

$$\begin{aligned} \Delta lnC(z^t) &= \theta \Delta lnY(z^t) + (1-\theta)\Delta lnY^*(z^t) \\ \Delta lnC^*(z^t) &= \theta \Delta lnY(z^t) + (1-\theta)\Delta lnY^*(z^t), \end{aligned}$$

With a symmetric assumption, $\sigma(\Delta lnY) = \sigma(\Delta lnY^*)$, where $\sigma(X)$ denotes the standard deviation of X. Let $\rho(X, X')$ denote the correlation between variables X and X'. Then, we can derive the consumption growth correlation as

$$\rho(\Delta lnC, \Delta lnC^*) = \frac{2\theta(1-\theta) + (\theta^2 + (1-\theta)^2)\rho(\Delta lnY, \Delta lnY^*)}{(\theta^2 + (1-\theta)^2) + 2\theta(1-\theta)\rho(\Delta lnY, \Delta lnY^*)}. \tag{5}$$

The parameter θ crucially governs the correlation of consumption growth in our model. If $\theta = 1$, then $\rho(\Delta lnC, \Delta lnC^*) = \rho(\Delta lnY, \Delta lnY^*)$. In this case, the preferences exhibit a complete home bias in consumption, and hence there is no goods trade between the two countries. If $\theta = 0.5$, then $\rho(\Delta lnC, \Delta lnC^*) = 1$. The perfect correlation in consumption growth arises when there is no home bias in consumption.

2.4 Leverage and Assets Supply

In each country, three types of assets are available: state-contingent bonds, risky equities and risk-free bonds. Both risky equities and risk-free bonds are claims to the diversifiable income. Equities represent a leveraged claim to aggregate diversifiable income. The leverage ratio is constant over time and denoted by ϕ. Let $\overline{B}_t(z^t)$ denote the supply of a one-period risk-free bond in period t in the home country and $W_t(z^t)$ denote the price of a claim to home country's aggregate diversifiable income in period t. With a constant leverage ratio, the total supply of $\overline{B}_t(z^t)$ must be adjusted such that

$$\overline{B}_t(z^t) = \phi \left[W_t(z^t) - \overline{B}_t(z^t) \right].$$

By the previous equation, the aggregate diversifiable output can be decomposed into the interest payment to bondholders and payouts to shareholders; the total payouts, including cash dividends and net repurchases, denoted $\overline{D}_t(z^t)$, are

$$\overline{D}_t(z^t) = (1-\alpha)Y(z^t) - R^f_{t,t-1}(z^{t-1})\overline{B}_{t-1}(z^{t-1}) + \overline{B}_t(z^t), \tag{6}$$

where $R^f_{t,t-1}(z^{t-1})$ denotes the home risk-free rate at period $t-1$. For simplicity, our model assumes that the supply of equity shares is constant. As a result, if a firm reissues or repurchases equity shares, it must be reflected by $\overline{D}_t(z^t)$ in our model. Similarly, the supply of foreign bonds is given by

$$\overline{B}^*_t(z^t) = \phi^* \left[W^*_t(z^t) - \overline{B}^*_t(z^t) \right],$$

while the payouts on foreign equity are given by

$$\overline{D}^*_t(z^t) = (1-\alpha^*)Y^*(z^t) - R^{*f}_{t,t-1}(z^{t-1})\overline{B}^*_{t-1}(z^{t-1}) + \overline{B}^*_t(z^t), \tag{7}$$

where $R^{*f}_{t,t-1}(z^{t-1})$ denotes the foreign risk-free rate at period $t-1$.

Finally, we denote the value of total home equity or a claim to total payouts on $\overline{D}_t(z^t)$ by $V_t(z^t)$. Likewise, we denote the value of total foreign equity or a claim to total payouts on $\overline{D}^*_t(z^t)$ by $V^*_t(z^t)$. The gross returns of home and foreign equities, or $R^d_{t,t-1}(z^t)$ and $R^{*d}_{t,t-1}(z^t)$, respectively, are therefore given by

$$R^{d}_{t,t-1}(z^t) = \frac{\overline{D}_t(z^t) + V_t(z^t)}{V_{t-1}(z^{t-1})} \tag{8}$$

$$R^{*d}_{t,t-1}(z^t) = \frac{\overline{D}^*_t(z^t) + V^*_t(z^t)}{V^*_{t-1}(z^{t-1})}. \tag{9}$$

2.5 Heterogeneity in Trading Technologies

There is significant portfolio heterogeneity not only across countries but also across investors within a country. To capture such heterogeneity, we implement the approach adopted by Chien, Cole, and Lustig (2011) and exogenously impose different restrictions on investors' portfolio choices. These restrictions apply to the menu of assets that these investors can trade as well as the composition of households' portfolios.

There are two classes of investors in terms of their asset trading technologies. The first class of investors faces no restrictions on portfolio choices and the menu of tradable assets. Specifically, these investors trade a complete set of contingent claims on the domestic and the foreign endowment. We call these investors Mertonian traders. They optimally adjust their portfolio choices in response to changes in the investment opportunity set. Hence, they act as currency market arbitrageurs and price exchange rate risk in our model.

The second class of investors faces restrictions on their portfolios and are called non-Mertonian traders. Specifically, their portfolio composition is restricted to be constant over time. We assume two types of non-Mertonian traders as follows: The first type are non-Mertonian equity investors, who can trade domestic equities and the domestic risk-free bonds. The other type are non-participants, who invest in only the domestic risk-free bonds. Even though the portfolio composition of non-Mertonian traders is exogenously given, they can still optimally choose how much to save.

Non-Mertonian equity investors deviate from the optimal portfolio choices in two dimensions. First, they cannot change the share of equities in their portfolios in response to changes in the market price of risk, which indicates missed market timing. Second, their portfolio share in equities might deviate from the optimal one on average.

We denote the fraction of different types of investors in the home country and the foreign country by μ_j and μ^*_j, where $j \in \{me, et, np\}$ represents Mertonian investors, non-Mertonian equity investors, and nonparticipants, respectively.

2.5.1 Trading of Mertonian Investors

We start by considering a version of our economy in which all trade occurs sequentially. Securities markets are segmented. Only the Mertonian traders have access to all securities markets. A home Mertonian trader who enters the period with net financial wealth $a_t(z^t, \eta^t)$ in node (z^t, η^t) has accumulated domestic claims worth $a_{ht}(z^t, \eta^{t-1}))$ and claims on foreign investments worth $a_{ft}(z^t, \eta^{t-1})$:

$$a_t(z^t, \eta^{t-1}) = a_{ht}(z^t, \eta^{t-1}) + \frac{a_{ft}(z^t, \eta^{t-1})}{e_t(z^t)},$$

where a_{ht} denotes the payoff of state-contingent claims in the home country expressed in terms of the home consumption basket, and a_{ft} denotes the payoff on foreign state contingent claims expressed in terms of the foreign consumption basket. Finally, e_t denotes the RER or the price of the home consumption basket relative to the foreign consumption basket.

$Q(z^{t+1}|z^t)$ and $Q^*(z^{t+1}|z^t)$ are the state-contingent prices in the home country and the foreign country, expressed in units of the home and foreign consumption basket respectively. The variables $q_n(z^t)$ and $q_x^*(z^t)$ denote the price of the home nontraded good in terms of the home consumption basket, and the price of the home export good in terms of the foreign consumption basket, respectively. At the end of the period, home Mertonian traders go to securities markets to buy domestic and foreign state-contingent claim $a_{h,t+1}(z^{t+1}, \eta^{t+1})$ and $a_{f,t+1}(z^{t+1}, \eta^{t+1})$, and they go to the goods market to purchase $c(z^t, \eta^t)$ units of the home consumption basket, subject to the following one-period budget constraint:

$$\begin{aligned} & \sum_{z_{t+1}} Q(z^{t+1}|z^t) a_{h,t+1}(z^{t+1}, \eta^t) + \sum_{z_{t+1}} Q^*(z^{t+1}|z^t) \frac{e_t(z^t)}{e_{t+1}(z^{t+1})} a_{f,t+1}(z^{t+1}, \eta^t) + c(z^t, \eta^t) \\ \leq \;\; & a_t(z^t, \eta^t) + \alpha \left[q_n(z^t) Y_n(z^t) + \frac{q_x^*(z^t)}{e_t(z^t)} Y_x(z^t) \right] \eta_t, \;\; for \;\; all \;\; (z^t, \eta^t). \end{aligned}$$

Note that the numeraire is the home consumption basket. Investors can spend all of their nondiversifiable income and with which the accumulated wealth they entered the period.

Similarly, the net financial wealth $a_t^*(z^t, \eta^t)$ of a foreign Mertonian trader at the start of the period consists of the net financial claims on the home endowment $a_{ht}^*(z^t, \eta^{t-1})$ and the claims on the foreign endowment $a^*_{ft}(z^t, \eta^t)$:

$$a^*_t(z^t, \eta^{t-1}) = a^*_{ht}(z^t, \eta^{t-1})e_t(z^t) + a^*_{ft}(z^t, \eta^{t-1}).$$

For the foreign households, their budget constraint is specified in terms of the foreign consumption basket. At the end of the period, the foreign Mertonian trader buys state-contingent claims $a^*_{ht}(z^t, \eta^t)$ and $a^*_{ft}(z^t, \eta^t)$ in financial markets and consumes $c^*(z^t, \eta^t)$ units of the foreign consumption basket. The flow budget constraint is given by

$$\begin{aligned} & \sum_{z_{t+1}, \eta_{t+1}} Q(z^{t+1}|z^t) a^*_{h,t+1}(z^{t+1}, \eta^{*t}) \frac{e_{t+1}(z^{t+1})}{e_t(z^t)} + \sum_{z_{t+1}} Q^*(z^{t+1}|z^t) a^*_{t+1}(z^{t+1}, \eta^{*t}) + c^*(z^t, \eta^{*t}) \\ \leq \ & a^*_t(z^t, \eta^{*,t-1}) + \alpha \left[q^*_n(z^t) Y^*_n(z^t) + q_x(z^t) e_t(z^t) Y^*_t(z^t) \right] \eta^*_t, \ \ for \ \ all \ \ (z^t, \eta^{*t}), \end{aligned}$$

where $q^*_n(z^t)$ and $q_x(z^t)$ denote the price of the foreign nontraded good in terms of the foreign consumption basket and the price of the foreign export good in terms of the home consumption basket, respectively. Note that all investors are subject to non-negative net wealth constraints, given by $a_t(z^t, \eta^t) \geq 0$ and $a^*_t(z^t, \eta^t) \geq 0$.

Currency Markets Arbitrageurs These Mertonian investors are currency arbitrageurs in our model. They enforce the no-arbitrage condition in the currency market, which governs the evolution of exchange rates:

$$Q(z^{t+1}|z^t) = Q^*(z^{t+1}|z^t) \frac{e_t(z^t)}{e_{t+1}(z^{t+1})}.$$

Taking the logarithm of the no-arbitrage condition yields the following:

$$\ln \frac{e_{t+1}}{e_t} = \ln Q_{t+1} - \ln Q^*_{t+1}. \tag{10}$$

Hence, the percentage rate of depreciation of the RER is determined by the percentage difference between the home and foreign pricing kernels.

2.5.2 Trading of Non-Mertonian Investors

The non-Mertonian investors are restricted to fixed portfolio weights. Their total asset holding at the beginning of period t, is given by their asset position at the end of the previous period, denoted by $\widehat{a}_{t-1}(z^{t-1}, \eta^{t-1})$, multiplied by the gross portfolio return, $R^{p}_{t,t-1}(z^t)$, which depends on their fixed portfolio. These non-Mertonian investors face the following budget constraint for all (z^t, η^t):

$$\widehat{a}_t(z^t, \eta^t) + c(z^t, \eta^t) \leq R^{p}_{t,t-1}(z^t)\widehat{a}_{t-1}(z^{t-1}, \eta^{t-1}) + \alpha \left[q_n(z^t)Y_n(z^t) + \frac{q^*_x(z^t)}{e_t(z^t)} Y_x(z^t) \right] \eta_t,$$

where all variables are expressed in units of the home consumption basket. The gross return on the fixed portfolio is given by

$$R^{p}_{t,t-1}(z^t, \eta^t) = \omega R^{d}_{t,t-1}(z^t) + (1-\omega) R^{f}_{t,t-1}(z^t),$$

where ω denotes the fixed portfolio shares in domestic equities. In the case of nonparticipants, ω is zero.

The budget constraint of the non-Mertonian investors in the foreign country is given by

$$\widehat{a}^*_t(z^t, \eta^{*t}) + c^*(z^t, \eta^{*t}) \leq R^{p,*}_{t,t-1}(z^t, \eta^{*t})\widehat{a}^*_{t-1}(z^{t-1}, \eta^{*,t-1}) + \alpha \left[q^*_n(z^t)Y^*_n(z^t) + q_x(z^t)e_t(z^t)Y^*_x(z^t) \right] \eta^*_t,$$

for all (z^t, η^{*t}). The gross return on the fixed portfolio is given by

$$R^{p,*}_{t,t-1}(z^t, \eta^t) = \omega^* R^{d*}_{t,t-1}(z^t) + (1-\omega^*) R^{f*}_{t,t-1}(z^t),$$

where ω^* denotes the fixed portfolio share in foreign equities. Finally, all investors are subject to nonnegative net wealth constraints, given by $\widehat{a}_t(z^t, \eta^t) \geq 0$ and $\widehat{a}^*_t(z^t, \eta^t) \geq 0$.

The details of the household problem and its associated Euler equations are described in Appendix A.1 and A.2.

2.6 Competitive Equilibrium

A competitive equilibrium for this economy is defined in the standard way. It consists of allocations of consumption, allocations of bond and equity choices, and a list of prices such that (i) given these prices, a trader's asset and consumption choices maximize her expected utility subject to the budget constraints, the nonnegative net wealth constraints, and the constraints on portfolio choices; and (ii) all asset markets clear.

2.6.1 Pricing Kernel

We use a recursive multiplier method to solve for equilibrium allocations and prices. This approach has the advantage that we can express the household consumption share, which is her consumption c relative to aggregate consumption C, in terms of a ratio of his recursive multiplier, ζ, to a single cross-sectional moment of the multiplier distribution, which we call h. To be precise,

$$\frac{c(z^t,\eta^t)}{C(z^t)} = \frac{\zeta(z^t,\eta^t)^{-\frac{1}{\gamma}}}{h_t(z^t)},$$

where $\zeta(z^t,\eta^t)$ is the recursive Lagrangian multiplier of the domestic household and $h_t(z^t)$ is defined as a $-1/\gamma$ moment of $\zeta(z^t,\eta^t)$ across traders. Please refer to Appendix A.3 for details. Similarly, the same consumption sharing rule is applied for foreign investors:

$$\frac{c^*(z^t,\eta^t)}{C^*(z^t)} = \frac{\zeta^*(z^t,\eta^t)^{-\frac{1}{\gamma}}}{h^*_t(z^t)}$$

As a result, the home stochastic discount factor is given by the standard Breeden-Lucas expression with a multiplicative adjustment:

$$Q_{t+1}(z^{t+1}|z^t) = \beta\left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}\left(\frac{h_{t+1}(z^{t+1})}{h_t(z^t)}\right)^{\gamma}\pi(z_{t+1}|z^t). \tag{11}$$

This can be interpreted as the intertemporal marginal rate of substitution (IMRS) of an unconstrained domestic Mertonian investor. The foreign stochastic discount factor of the foreign country is given by

$$Q^*_{t+1}(z^{t+1}|z^t) = \beta \left(\frac{C^*(z^{t+1})}{C^*(z^t)} \right)^{-\gamma} \left(\frac{h^*_{t+1}(z^{t+1})}{h^*_t(z^t)} \right)^{\gamma} \pi(z_{t+1}|z^t); \tag{12}$$

this can be interpreted as the IMRS of an unconstrained foreign Mertonian investor. As a result, the percentage rate of depreciation of the RER is given by

$$\Delta \ln e_{t+1} = -\gamma(\Delta \ln C_{t+1} - \Delta \ln C^*_{t+1}) + \gamma(\Delta \ln h_{t+1} - \Delta \ln h^*_{t+1}).$$

2.6.2 Segmentation Mechanism

The key mechanism of our model works through the concentration of global aggregate risk and country-specific endowment risk among a small pool of investors. As we show later, our calibrated model features a large pool of non-Mertonian investors whose exposure to both global risk and country-specific risk is relatively low, given their low-risk and home-biased portfolios. In contrast, a relatively small set of investors equipped with better trading technologies – the so-called Mertonian investors – can accumulate a higher level of wealth and smooth consumption better. More importantly, their sophisticated trading technologies allow them to share the country-specific risk with foreign investors.

The heterogeneity of portfolio choices has a great impact on pricing. First, the concentration of aggregate risk among a small group of Mertonian investors generates a high market price of risk. Moreover, the equity home bias among non-Mertonian investors offers a good risk-sharing opportunity among Mertonian investors across countries. Given the fact that Mertonian investors are marginal traders who price risk, the pricing kernels could be highly correlated as a result of sharing country-specific risk among them, despite the limited sharing capacity at the aggregate level. In particular, the risk sharing at the aggregate level is restricted by a high degree of home bias in consumption.

2.7 A Special Case: The Two-Period Model

To demonstrate the mechanism in our model, this subsection describes a special case with two periods and the following simplified assumptions. First, the home and foreign countries are fully symmetric. Second, the endowment in period 1 is nondiversifiable, freely traded, not subject to uncertainty, and normalized to one unit of the consumption good; therefore, the exchange rate is unity in period 1. Third, the realization of each country's endowment in period 2 is either $1+g$ in the high-endowment state or $1-g$ in the low-endowment state, where $g>0$. Hence, there are four possible combinations of home and foreign endowments in period 2: $(Y_2(z_2), Y_2^*(z_2)) \in \{(1+g, 1+g), (1+g, 1-g), (1-g, 1+g), (1-g, 1-g)\}$, and $z_2 \in \{z_2^{hh}, z_2^{hl}, z_2^{lh}, z_2^{ll}\}$, where the superscript denotes the state of home and foreign endowments. Finally, there are only two types of households in each country: Mertonian traders and nonparticipants. We continue to use the superscripts me and np to represent the Mertonian traders and the nonparticipants, respectively. Both types are assumed to have the same initial wealth and endowment in the first period, denoted by Y_1 and W_1, respectively. The notations for the two-period model follow those in the previous subsections except that the time subscripts are restricted to $t=1,2$.

First, consider the budget constraints for the home Mertonian traders:

$$
\begin{aligned}
Y_1 + W_1 &= c_1^{me} + a_2^{me}, \\
c^{me}(z_2) &= \alpha \left[q_n(z_2) Y_n(z_2) + \frac{q_x^*(z_2)}{e(z_2)} Y_x(z_2) \right] + R_2^p(z_2) a_2^{me},
\end{aligned}
$$

where $R_2^p(z)$ is the portfolio return of Mertonian investors.

Next, the budget constraints for the home nonparticipants are

$$
\begin{aligned}
Y_1 + W_1 &= c_1^{np} + a_2^{np}, \\
c^{np}(z_2) &= \alpha \left[q_n(z_2) Y_n(z_2) + \frac{q_x^*(z_2)}{e(z_2)} Y_x(z_2) \right] + R_2^f a_2^{np},
\end{aligned}
$$

where the risk-free rate, R_2^f, is the return for nonparticipants.

2.7.1 Volatility and Correlation of Pricing Kernels

The second-period budget constraints suggest that both types of traders have the same exposure to the nondiversifiable income risk. However, the diversifiable income of the non-Mertonian traders is constant across the state, which has zero exposure to the aggregate risk. Therefore, the second-period consumption of the Mertonian traders has higher aggregate risk exposure than that of the nonparticipants. Hence, the Mertonian traders have more volatile consumption growth than the nonparticipants. The concentration of risk among Mertonian traders is the main mechanism generating the volatile pricing kernels.

Specifically, first let us consider the two aggregate states, z_2^{hh} and z_2^{ll}. In these two states, both countries are symmetric, thus the RER must be 1: $e(z_2^{hh}) = e(z_2^{ll}) = 1$. In other words, there is no exchange rate risk in these two states. The non-diversifiable income, which is the same for both types of traders, can be simplified to:

$$\alpha \left[q_n(z_2) Y_n(z_2) + \frac{q_x^*(z_2)}{e(z_2)} Y_x(z_2) \right] = \alpha \left[\theta + \frac{(1-\theta)}{e(z_2)} \frac{C^*(z_2)}{C(z_2)} \right] C(z_2) = \alpha C(z_2),$$

where the first equality uses the property that the expenditure shares on the export good and the nontraded good are θ and $1-\theta$, respectively (see Appendix A.4 for details).

As a result, the nonparticipants' consumption share in period 2 in these two states becomes

$$\frac{c^{np}(z_2^{hh})}{C(z_2^{hh})} = \alpha + \frac{R_2^f a_2^{np}}{C(z_2^{hh})} \text{ and } \frac{c^{np}(z_2^{ll})}{C(z_2^{ll})} = \alpha + \frac{R_2^f a_2^{np}}{C(z_2^{ll})}$$

Given $C(z_2^{hh}) > C(z_2^{ll})$, the consumption share of the nonparticipants is higher in the bad state, z_2^{ll}, than that in the good state, z_2^{hh}. To clear the market, the consumption share of Mertonian traders has to be the opposite. This implies that the nonparticipants bear less aggregate risk compared with the Mertonian traders. Therefore, the Mertonian traders' consumption in period 2 must be higher in z_2^{hh} and lower in z_2^{ll}. This is why the pricing kernel, which depends on the Mertonian traders' consumption growth, is volatile.

In the remaining aggregate states, z_2^{hl} and z_2^{lh}, these two countries have opposite direction of consumption growth rates unless $\theta = 0.5$. If the total income of nonparticipants has zero exposure to the aggregate risk, then the consumption growth correlation among the Mertonian traders have to be negative in these two states, which can significantly lower the correlation of the pricing kernels. Hence, some degree of aggregate risk exposure for the nonparticipants' nondiversifiable income is essential for weakening the negative correlation of the pricing kernels in these two states. To see why, consider the state z_2^{hl} as an example. In this state, the consumption of the nonparticipants is given by

$$c^{np}(z_2^{hl}) = \alpha \left[\theta C(z_2^{hl}) + \frac{(1-\theta)}{e(z_2^{hl})} C^*(z_2^{hl}) \right] + R_2^f a_2^{np},$$

which tends to increase (with $\theta > 0.5$) because of the aggregate risk exposure on the nondiversifiable income. Note that the Mertonian traders still load up aggregate risk in this state, while they can share risk with others by trading state-contingent claim across countries. Their risk-sharing behaviors also mitigate the movement of the RER and increase the consumption growth correlation among Mertonian traders across countries.

3 Quantitative Results

We calibrate our model to evaluate the extent to which our model can account for the international correlation in pricing kernels, the volatility of the pricing kernel and the volatility of the RER seen in the data. Our benchmark model considers a symmetric two-country model in which both countries have identical preferences, portfolio restrictions, and shock processes. The benchmark model is calibrated to match several key features of data, including the data on trade in goods and assets. We then perform a number of counterfactual exercises to examine the effects of a home bias in consumption on the dynamics behaviors of the RER and asset pricing.

In subsection 3.4, we demonstrate the effects of a home bias in consumption by varying the parameter θ. In addition, in subsection 3.5, we consider changes in the trader pool to highlight the role of equity market participation. The last subsection removes the heterogeneity of trading technologies – while keeping the home bias in consumption – to emphasize the importance of different trading technologies.

3.1 Calibration

The home country in our model is set to the United States. The foreign country is an aggregation of four countries: France, Germany, Japan, and the United Kingdom. We collect annual data from International Financial Statistics from 1980 to 2012. The share of U.S. gross domestic product (GDP) in our hypothetical world economy is on average 52%, which is close to half. Thus, we assume an equal size for home and foreign economies. For simplicity and the demonstration of our mechanism, we set parameters such that the two economies are fully symmetric. Given that condition, all the parameters we calibrate applied to both countries.

First, we set $1 - \theta$, or the home bias parameter, to half of our hypothetical world's ratio of trade to GDP, which is 0.16. Given θ, the consumption growth process in each country is pinned down by the endowment process because of the preference assumption. The innovation terms in the output shock process, or ε and ε^*, are calibrated into a four-state first-order Markov chain to match the following statistics: (1) The consumption growth correlation between two countries is 0.172 (2) The average consumption growth of each country is 2.13% with a standard deviation of 2.36% and (3) ρ is set to 0.95. The resulting correlation of home and foreign endowment growth is -0.21.

We also consider a two-state first-order Markov chain for idiosyncratic shocks. The first state is low and the second state is high. Following Storesletten, Telmer, and Yaron (2004), we calibrate this shock process by two moments: the standard deviation of idiosyncratic shocks and the first-order autocorrelation of shocks, except that we eliminate the countercyclical variation in idiosyncratic risk. The Markov process for the log of the nondiversifiable income share, or log η, has a standard deviation of 0.71 and its autocorrelation is 0.89. The transition probability is denoted by

$$\pi(\eta'|\eta) = \begin{bmatrix} 0.9450 & 0.0550 \\ 0.0550 & 0.9450 \end{bmatrix}.$$

The two states of the idiosyncratic shocks, whose mean is normalized to 1, are $\eta_L = 0.3894$ and $\eta_H = 1.6106$.

Following Mendoza, Quadrini, and Rios-Rull (2009), the fraction of nondiversifiable output is set to 88.75%. As shown in Section 2, equities in our model are simply leveraged claims to diversifiable income. Following Abel (1999) and Bansal and Yaron (2004), the leverage ratio parameter is set to 3.

The model operates at an annual frequency. We set the time discount factor $\beta = 0.95$ to deliver the low risk-free rate. The risk-aversion rate γ is set to 5.5 to help to produce a high risk premium in our benchmark calibration.

To match a high market price of risk, a small fraction of Mertonian traders must absorb a large amount of aggregate risk. We therefore set the fraction of Mertonian traders to 5% for both countries. Since 50% of U.S. investors do not hold stocks (according to the 2010 Survey of Consumer Finance data), we set 50% of investors as nonparticipants. The remaining investors are non-Mertonian equity investors, and they represent 45%. We set their portfolio shares of home equities and the home risk-free bonds to 25% and 75%, respectively.

3.2 Computation

The equilibrium RER in our model is stationary. Thanks to the assumption of symmetry, the endowment processes of both countries share the same trend. Therefore, the ratio of aggregate consumption between these two countries is stationary. Since the consumption shares of each type of trader are bounded above and below, the ratio of consumption of home to foreign domestic traders has to be finite and stationary. The RER is also equal to the relative marginal utility of consumption between home and foreign unconstrained Mertonian investors. Therefore, the RER is stationary if there is a nonzero measure of non-binding Mertonian traders for both countries in every possible state, because the relative consumption share of home and foreign Mertonian investors is stationary.

Given that the RER e_t is stationary, it cannot depend on the entire aggregate history z^t. Importantly, h and h^* do depend on the entire aggregate history, but the log difference, which determines the RER, has to be stationary. In other words, h and h^* have to share the same stochastic trend for the RER to be stationary.

We use $z_k \in \mathcal{A}$ as summary statistics for the aggregate history in each country. We assume that the level of the RER depends only on the summary statistics z_k. The RER at a grid point $z_k \in \mathcal{A}$ can be derived from the pricing kernels as follows:

$$e_t(z_k) = \frac{P(z_k)}{P^*(z_k)} = \frac{C(z_k)^{-\gamma} h(z_k)^{\gamma}}{C^*(z_k)^{-\gamma} h^*(z_k)^{\gamma}} = \left(\frac{m_t^*(z_k)}{m_t(z_k)}\right)^{-\gamma(1-2\theta)} \frac{h(z_k)^{\gamma}}{h^*(z_k)^{\gamma}},$$

where P and P^* denote for time-zero prices for home and foreign consumption, respectively (see Appendix A for details).

We define $\ln(\frac{h(z_k)}{h^*(z_k)}) \equiv b(z_k)$. Given the stationarity condition, the RER can be written only as a function of the summary statistics for aggregate history:

$$\ln e(z_k) = \gamma(2\theta - 1) \ln\left(\frac{m^*(z_k)}{m(z_k)}\right) + \gamma b(z_k) \tag{13}$$

In our computation, we record the growth rate of h and h^*. Given the stationarity of e_t, the innovation in h at home and abroad has to satisfy

$$\ln\left(\frac{h(z_k')}{h(z_k)}\right) - \ln\left(\frac{h^*(z_k')}{h^*(z_k)}\right) = b(z_k') - b(z_k) \equiv \Delta(z_k', z_k). \tag{14}$$

Given $\{b(z_k),\ \ln\left(\frac{h(z_k')}{h(z_k)}\right), \ln\left(\frac{h^*(z_k')}{h^*(z_k)}\right)\}$, we can completely characterize an equilibrium of this economy, because we have the equilibrium prices, the RERs, and the allocations. See Appendix B for details.

3.3 Quantitative Results of the Benchmark Case

We report the model statistics for the benchmark case in Table I. Taking a standard deviation of equation (10) together with the assumption of symmetric countries, the standard deviation of RER depreciation in our model is determined by the moments of pricing kernels as follows:

$$\sigma(\Delta \ln e_{t+1}) = \sigma(\ln Q)\sqrt{2(1-\rho(\ln Q, \ln Q^*))}. \tag{15}$$

According to (15), the standard deviation of RER depreciation is increasing in the standard deviation of the pricing kernels and decreasing in the international correlation in the pricing kernels.

Our benchmark economy delivers high volatility and high correlation of pricing kernels, and these statistics are extremely close to the observed statistics. The standard deviation of the pricing kernels in the model is 0.42 and the correlation of the pricing kernels is 97.5%. According to (15), these statistics imply that the standard deviation of RER depreciation is 9.4%, whereas the observed volatility is 13%. Hence, our calibrated model is capable of producing reasonable volatility of the RER and the pricing kernels, despite the low international correlation in aggregate consumption.

The success of matching the pricing data under very limited consumption risk sharing relies on two mechanisms governing the two moments in (15). The first mechanism is the uneven distribution of aggregate risk and country-specific risk across population. The uneven distribution of risk is caused by investors' heterogeneous portfolio choices. Given that the non-Mertonian investors bear only a relatively small share of the aggregate risk, most of the aggregate risk is borne by Mertonian investors. The concentration of risk among a small set of investors leads to high volatility of the pricing kernels. The second mechanism relies on the ability of the Mertonian investors in the two countries to bear aggregate risk and share country-specific risk among themselves. Therefore, their consumption tends to synchronize, which produces highly correlated pricing kernels.

Also, as a group consumption and portfolio choices of the Mertonian investors are less restricted by the presence of nontraded consumption than those of the non-Mertonian investors because they represent only a small share of the population. However, if the international trade in goods is completely shut down, then there is no reasons for Mertonian investors to hold any external asset. We demonstrate in the next subsection that international trade is essential for the Mertonian investors to share risk across countries.

In Table I, last three rows shows the moments of consumption by investor group. The correlation of consumption and aggregate consumption for the non-Mertonian traders is almost perfect, 0.975, and the corresponding correlation for the Mertonian traders is much lower, 0.725. The correlation of consumption and income is higher for the non-Mertonian equity traders. We can compare this ranking with the estimates of income elasticity of demand in Hummels and Lee (2013). These authors use the U.S. consumer expenditure survey (CEX) data from 1994 to 2010 to estimate the income elasticity of demand for export goods for each percentile of income distribution. They find that the investors' income elasticity of demand for export goods is decreasing in household income. In our model, the income elasticity of demand for export goods is identical to the income elasticity of demand for final consumption. Therefore, our ranking of the income elasticity of demand will be consistent with theirs if the Mertonian traders have higher income than the non-Mertonian equity traders. This is the case in our model, because the Mertonian traders accumulate a larger amount of high-return assets than the non-Mertonian equity traders.

The last row in Table I shows the standard deviation of the Mertonian traders' consumption relative to the standard deviation of the non-Mertonian equity traders consumption. This ratio in our model is 3.970, indicating that the Mertonian traders' consumption is more volatile. There are two reasons for this. First, the Mertonian traders share aggregate risk among themselves, while the non-Mertonian equity traders do not. Second, the idiosyncratic income shocks do not matter to volatility of aggregate consumption within the group, thanks to the assumption of the law of large numbers. The ranking of consumption volatility in our model is consistent with the evidence in Parker and Vissing-Jorgensen (2009). Using the CEX survey data from 1982 to 2004, they find that the top 5% of investors are estimated to be about 4.5 times more exposed to aggregate consumption shocks than those in the bottom 80%.

[Table 1 about here.]

3.4 Impacts of Home Bias in Consumption

In this subsection, we investigate the role of a home bias in consumption. Intuitively, without a home bias in consumption and without the nontraded good, the law of one price holds and the RER is constant. In this case, investors achieve full risk sharing through international goods markets regardless of frictions in financial markets. To the contrary, if there is no trade due to either a complete home bias or prohibitively large trade frictions, then there will be no incentives for investors to hold external assets.

To explore the impacts of a home bias in consumption on the RER volatility, we consider two exercises. First, we increase the share of the home goods in final consumption expenditure from 0.84 to 0.95, which significantly reduces the volume of trade in goods. The second column of Table II shows these results. There is virtually no change in the volatility of pricing kernel, while the correlation of pricing kernels drops from 0.975 to 0.892.

A higher degree of home bias in consumption implies that consumers are less inclined to consume the foreign good and hence reduces the incentive to share the country-specific risk with foreign consumers. As a result, the international correlation in consumption growth falls from 16.9% to −10.9%. The fall in the correlation of consumption growth can also be understood by equation (5). As θ approaches unity, the correlation of consumption growth approaches to the correlation of endowment growth, which is −21%.

Evidently, a high degree of home bias in consumption significantly reduces the correlation in the pricing kernels, even though only a small fraction of investors are sharing the country-specific risk. Given equation (15), conditioning on unchanged volatility of the pricing kernels, the sharp fall in the correlation in the pricing kernels produces a sharp increase – as much as 20% – in the RER volatility. The magnitude is more than double that in the benchmark case. Such a positive impact of a home bias in consumption on the RER volatility is similar to the model by Warnock (2003), in which the RER is volatile as a result of nominal shocks.

The second exercise involves the removal of home bias in consumption by setting the share of the nontraded goods in consumption expenditure at 0.5. The third column of Table II confirms that without a home bias in consumption the RER is not volatile, although the pricing kernel is as volatile as in the benchmark case. In this case, the domestic pricing kernel is perfectly correlated with the foreign pricing kernel, as predicted by equation (5), and consequently international risk sharing is complete. Therefore, according to (15) the perfect correlation in pricing kernels produces zero standard deviation of the RER.

These two exercises demonstrate that the home bias in consumption is necessary for generating RER volatility, although a too-high degree of home bias in consumption can generate higher RER volatility than the data. In other words, international trade in goods is critical for international risk sharing. This insight from our model is novel, compared with the existing models, that explain RER volatility as a result of international trade in assets without international trade in goods, such as Alvarez, Atkeson, and Kehoe (2002) and Colacito and Croce (2011). Moreover, our finding is supported by the recent empirical evidence of Fitzgerald (2012). She finds that trade costs impede risk sharing among developed countries, but financial frictions do not impede risk sharing among them. Her finding suggests that international trade in goods is necessary for international risk sharing, as in our model.

[Table 2 about here.]

3.5 Changes in Composition of Traders

In this subsection, we vary the composition of traders to examine the impacts of distribution of aggregate risk on the volatility of the RER, that of the pricing kernels, the international correlation of pricing kernels, and that of consumption growth.

First, we vary equity market participation rate by changing the relative size of population between the nonparticipants and the non-Mertonian equity traders, while keeping the size of the Mertonian traders constant. In columns (2) and (3) of Table III, we reduce the size of the non-participants from 50% to 40% and 30%, respectively. The results show that, as the equity market participation rate increases, country-specific risk becomes less concentrated among the Mertonian traders. Thus, the volatility of pricing kernel falls as the the equity market participation rate increases. A 10% increase in the equity market participation rate reduces the standard deviation of the pricing kernels by roughly 3% to 4%.

Since the equity portfolio held by the new equity traders is biased toward the domestic equity, their increasing participation forces the Mertonian traders to bear relatively higher residual risk from abroad. As a result, the Mertonian traders increase risk sharing across borders and, as a result, the correlation of the pricing kernels increases. A 10% increase in the equity market participation rate raises the correlation by 0.1% to 0.2%. Overall, according to equation (15), both the increase in the correlation of the pricing kernels and the decrease in their volatility reduce the RER volatility. Quantitatively, a 10% increase in the equity market participation rate reduces the standard deviation of the RER by roughly 1%.

Next, we vary the composition of the equity market participants, holding the population size of nonparticipants unchanged. In columns 4-5 of Table III, we increase the population size of the Mertonian traders from 5% to 10% and then to 20%, respectively. As the size of the Mertonian traders increases, aggregate risk becomes less concentrated among them. Thus, the volatility of the pricing kernels falls. As the size of the Mertonian traders increases from 5% to 10% and 10% to 20%, the standard deviation of the pricing kernels falls 7.9% and 8.2%, respectively.

In addition, an increase in the size of the Mertonian traders reduces the residual risk that is shared with their foreign counterparts; therefore the correlation of pricing kernels decreases. As we increase the size of the Mertonian traders in columns (4) and (5), the correlation of the pricing kernels falls from 97.5% to 95.7% and to 90.5%, respectively.

According to equation (15), the decrease in the correlation of the pricing kernels and the decrease in their volatility have competing effects on the RER volatility. Quantitatively, the decrease in the correlation of the pricing kernels dominates and the RER volatility increases as a result of an increase in the size of the Mertonian traders.

[Table 3 about here.]

Finally, the last column in Table III explores the impacts of financial deepening by assuming that all investors are Mertonian traders. Hence, all investors respond to the change in investment opportunities in every period by optimally adjusting their portfolios and make no investment mistakes. Compared with the benchmark case, the standard deviation of RER in column (6) is significantly reduced to 0.131. The reason is that aggregate risk is equally distributed over the entire population, especially within a country. Consequently, there is little residual risk to share across borders. For this reason, the correlation of the pricing kernels falls significantly from 0.975 to only 0.387.

4 Conclusion

We use a general equilibrium model with asset trading restrictions and consumption home bias to demonstrate that RER volatility is related to frictions in both goods and financial markets. The asset trading restrictions imposed in our model are in line with the empirical evidence in the household finance literature. With a realistic assumption that most investors do not actively participate in the domestic and foreign equity markets, we reconcile highly correlated and volatile pricing kernels with low correlation in consumption growth.

The insight from our model is that the high cross-country correlation in the pricing kernels is not necessarily evidence of a high degree of international risk sharing. In particular, international risk sharing is aggressively undertaken by a small fraction of sophisticated investors facing no restrictions on asset trade. Despite the small size of fraction, these investors are the currency arbitrageurs and their portfolio adjustment determines RER volatility. In fact, their portfolio adjustment generates a positive correlation between the RER and relative consumption growth, as in equation (10). Hence, our model has not solved the Backus-Smith puzzle, which documents a negative correlation between the RER and relative consumption growth.

References

Abel, A. B. (1999): "Risk Premia and Term Premia in General Equilibrium," *Journal of Monetary Economics*, 43, 3–33.

Alvarez, F., A. Atkeson, and P. J. Kehoe (2002): "Money, Interest Rates, and Exchange Rates with Endogenously Segmented Markets," *Journal of Political Economy*, 110(1), 73–112.

Bansal, R., and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Prizing Puzzles," *Journal of Finance*, 59, 1481–1509.

Brandt, M. W., J. H. Cochrane, and P. Santa-Clara (2006): "International Risk Sharing Is Better Than You Think, or Exchange Rates Are Too Smooth," *Journal of Monetary Economics*, 53(4), 671–698.

Chien, Y., H. Cole, and H. Lustig (2011): "A Multiplier Approach to Understanding the Macro Implications of Household Finance," *Review of Economic Studies*, 78(1), 199–234.

Coeurdacier, N., and H. Rey (2013): "Home Bias in Open Economy Financial Macroeconomics," *Journal of Economic Literature*, 51(1), 63–115.

Colacito, R., and M. M. Croce (2011): "Risks for the Long Run and the Real Exchange Rate," *Journal of Political Economy*, 119(1), 153–181.

Farhi, E., and X. Gabaix (2008): "Rare Disasters and Exchange Rates," NBER Working Paper 13805.

Fitzgerald, D. (2012): "Trade Costs, Asset Market Frictions, and Risk Sharing," *American Economic Review*, 102(6), 2700–2733.

Guiso, L., and P. Sodini (2012): "Household Finance: An Emerging Field," CEPR Discussion Papers.

Hummels, D. L., and K.-Y. Lee (2013): "Income Elastic Goods and the Great Trade Collapse: Evidence from MicroData," *Working Paper, Purdue University.*

Lustig, H., and A. Verdelhan (2015): "Does Market Incompleteness Help to Understand Exchange Rates?," Working Paper UCLA Anderson School of Management.

Mendoza, E. G., V. Quadrini, and J.-V. Rios-Rull (2009): "Financial Integration, Financial Development, and Global Imbalances," *Journal of Political Economy*, 117(3), 371–416.

Parker, J. A., and A. Vissing-Jorgensen (2009): "Who Bears Aggregate Fluctuations and How?," *American Economic Review Papers and Proceedings*, 99(2), 399–405.

Pavlova, A., and R. Rigobon (2010): "International Macro-Finance," NBER Working Papers 16630, National Bureau of Economic Research.

Stathopoulos, A. (2012): "Asset Prices and Risk Sharing in Open Economies," Working paper, University of Southern California.

Storesletten, K., C. I. Telmer, and A. Yaron (2004): "Cyclical Dynamics in Idiosyncratic Labor Market Risk," *Journal of Political Economy*, 112(3), 695–717.

Warnock, F. E. (2003): "Exchange Rate Dynamics and the Welfare Effects of Monetary Policy in a Two-Country Model with Home-Product Bias," *Journal of International Money and Finance*, 22(3), 343–363.

Table I: Benchmark Results

	Benchmark	Data
$\sigma(\ln Q) = \sigma(\ln Q^*)$	0.423	0.40
$\sigma(\Delta \ln e)$	0.094	0.13
$\rho(\ln Q, \ln Q^*)$	0.975	0.95
$\rho(\Delta \ln C, \Delta \ln C^*)$	0.169	0.171
$\rho(\Delta \ln c, \Delta \ln C)_{Mertonian}$	0.725	Low
$\rho(\Delta \ln c, \Delta \ln C)_{Non-Mertonian}$	0.975	High
$\frac{\sigma(\Delta \ln c)_{Mertonian}}{\sigma(\Delta \ln c)_{Non-Mertonian}}$	3.970	4.5

The simulation results are based on $18,000$ agents for each type and $10,000$ periods.

Table II: Results of Variation in Home Bias in Consumption

	Benchmark: $\theta = 0.84$	$\theta = 0.95$	$\theta = 0.5$
E(Trade/GDP)	0.32	0.10	1.00
$\sigma(\ln Q) = \sigma(\ln Q^*)$	0.423	0.435	0.419
$\sigma(\Delta \ln e)$	0.094	0.202	0
$\rho(\ln Q, \ln Q^*)$	0.975	0.892	1
$\rho(\Delta \ln C, \Delta \ln C^*)$	0.169	-0.109	1

The simulation results are based on $18,000$ agents for each type and $10,000$ periods.

Table III: Results of Variations in the Trader Pool

	(1) Benchmark	(2)	(3)	(4)	(5)	(6)
Population size of investors						
Mertonian	0.05	0.05	0.05	0.10	0.20	1.00
Non-Mertonian	0.45	0.55	0.65	0.40	0.30	0
Nonparticipants	0.50	0.40	0.30	0.50	0.50	0
$\sigma(\ln Q) = \sigma(\ln Q^*)$	0.423	0.396	0.360	0.344	0.262	0.119
$\sigma(\Delta \ln e)$	0.094	0.087	0.076	0.101	0.115	0.131
$\rho(\ln Q, \ln Q^*)$	0.975	0.976	0.978	0.957	0.905	0.387
$\rho(\Delta \ln C, \Delta \ln C^*)$	0.169	0.169	0.169	0.169	0.169	0.169

The simulation results are based on $18,000$ agents for each type and $10,000$ periods.

A Time-Zero Trading Household Problem

A.1 Time-Zero Trading

We describe an equivalent version of this economy in which all households trade at time zero. The time-zero price of a claim that pays one unit of consumption in node z^t can be constructed recursively from the one-period-ahead Arrow prices:

$$P(z^t)\pi(z^t) = Q(z_t|z^{t-1})Q(z_{t-1}|z^{t-2})...Q(z_1|z^0)Q(z_0),$$

$$P^*(z^t)\pi(z^t) = Q^*(z_t|z^{t-1})Q^*(z_{t-1}|z^{t-2})...Q^*(z_1|z^0)Q^*(z_0).$$

The real exchange rate is the ratio of Arrow-Debreu prices in node z^t:

$$e_t(z^t) = \frac{P(z^t)}{P^*(z^t)}.$$

The net financial wealth position of any trader in the home country given the history can be stated as

$$-a_t(z^t,\eta^t) = \sum_{s\geq t} \sum_{(z^s,\eta^s)\succeq(z^t,\eta^t)} \widetilde{P}(z^s,\eta^s)\left[\alpha\left(\frac{P_n(z^s)}{P(z^s)}Y_n(z^s) + \frac{P_x^*(z^s)}{P(z^s)}Y_x(z^s)\right)\eta_s - c(z^s,\eta^s)\right],$$

where $\widetilde{P}_n(z^t,\eta^t) = \pi(z^t,\eta^t)P_n(z^t)$. Similarly, the asset position of any foreign trader is

$$-a_t^*(z^t,\eta^t) = \sum_{s\geq t} \sum_{(z^s,\eta^s)\succeq(z^t,\eta^t)} \widetilde{P}(z^s,\eta^s)\left[\alpha\left(\frac{P_n^*(z^s)}{P^*(z^s)}Y_n^*(z^s) + \frac{P_x(z^s)}{P^*(z^s)}Y_x^*(z^s)\right)\eta_s - c^*(z^s,\eta^s)\right],$$

where $\widetilde{P}_n^*(z^t,\eta^t) = \pi(z^t,\eta^t)P_n^*(z^t)$. From the above equation, we are able to write the household problem in the form of time-zero trading fashion as shown in the next subsection.

A.2 Household Optimization Problem

Following Chien, Cole, and Lustig (2011), we state the household problem in this Arrow-Debreu economy.

A.2.1 Mertonian Traders

We start with the Mertonian traders' problem in the home country. There are two constraints. Let χ denote the multiplier on the present value budget constraint and $\varphi(z^t, \eta^t)$ denote the multiplier on debt constraints. The saddle-point problem of a Mertonian trader can be stated as follows:

$$
\begin{aligned}
L \;=\; & \min_{\{\chi,\nu,\varphi\}} \max_{\{c,a\}} \sum_{t=1}^{\infty} \beta^t \sum_{(z^t,\eta^t)} u(c(z^t,\eta^t))\pi(z^t,\eta^t) \\
& + \chi \left\{ \sum_{t=1}^{\infty} \sum_{(z^t,\eta^t)} \widetilde{P}(z^t,\eta^t) \left[\alpha \left(\frac{P_n(z^t)}{P(z^t)} Y_n(z^t) + \frac{P_x^*(z^t)}{P(z^t)} Y_x(z^t) \right) \eta_t - c(z^t,\eta^t) \right] + a_0(z^0) \right\} \\
& - \sum_{t=1}^{\infty} \sum_{(z^t,\eta^{h,t})} \varphi_t(z^t,\eta^t) \left\{ \sum_{s \geq t} \sum_{(z^s,\eta^{h,s}) \succeq (z^t,\eta^{h,t})} \widetilde{P}(z^s,\eta^s) \left[\alpha \left(\frac{P_n(z^s)}{P(z^s)} Y_n(z^s) + \frac{P_x^*(z^s)}{P(z^s)} Y_x(z^s) \right) \eta_s - c(z^s,\eta^s) \right] \right\}.
\end{aligned}
$$

The first-order condition with respect to consumption is given by

$$\beta^t u'(c(z^t,\eta^t)) = \zeta(z^t,\eta^t) P_x(z^t) \;\; for \;\; all \;\; (z^t,\eta^t), \tag{16}$$

where $\zeta(z^t, \eta^{i,t})$ is defined recursively as

$$\zeta_t(z^t,\eta^t) = \zeta_{t-1}(z^{t-1},\eta^{t-1}) - \varphi_t(z^t,\eta^t),$$

with initial $\zeta_0 = \chi$. It is easy to show that this is a standard convex constraint maximization problem. Therefore, the first-order conditions are necessary and sufficient.

A.2.2 Non-Mertonian Traders

Non-Mertonian traders face additional restrictions on their portfolio choices. Let $\nu_t(z^t, \eta^t)$ denote the multiplier on portfolio restrictions. Given the same definition of other multipliers as in the active trader problem, the saddle-point problem of a nonparticipant trader whose asset in the end of the period is $\widehat{a}_{t-1}(z^{t-1}, \eta^{t-1})$ in the home country can be stated as

$$\begin{aligned}
L \;=\; & \min_{\{\chi,\nu,\varphi\}} \max_{\{c,\widehat{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{(z^t,\eta^t)} u(c_t(z^t,\eta^t))\pi(z^t,\eta^t) \\
& \chi \left\{ \sum_{t=1}^{\infty} \sum_{(z^t,\eta^t)} \widetilde{P}(z^t,\eta^t) \left[\alpha \left(\frac{P_n(z^t)}{P(z^t)} Y_n(z^t) + \frac{P_x(z^t)}{P(z^t)} Y_x(z^t) \right) \eta_t - c(z^t,\eta^t) \right] + a_0(z^0) \right\} \\
& + \sum_{t=1}^{\infty} \sum_{(z^t,\eta^{h,t})} \nu_t(z^t,\eta^t) \left\{ \begin{array}{c} \sum_{s\geq t} \sum_{(z^s,\eta^s)\succeq(z^t,\eta^t)} \widetilde{P}(z^s,\eta^s) \left[\alpha \left(\frac{P_n(z^s)}{P(z^s)} Y_n(z^s) + \frac{P_x^*(z^s)}{P(z^s)} Y_x(z^s) \right) \eta_s - c(z^s,\eta^s) \right] \\ + \widetilde{P}(z^t,\eta^t) R^p_{t,t-1}(z^t) a e^i_{t-1}(z^{t-1},\eta^{t-1}) \end{array} \right\} \\
& - \sum_{t=1}^{\infty} \sum_{(z^t,\eta^{h,t})} \varphi_t(z^t,\eta^t) \left\{ \sum_{s\geq t} \sum_{(z^s,\eta^s)\succeq(z^t,\eta^t)} \widetilde{P}(z^s,\eta^s) \left[\alpha \left(\frac{P_n(z^s)}{P(z^s)} Y_n(z^s) + \frac{P_x^*(z^s)}{P(z^s)} Y_x(z^s) \right) \eta_s - c(z^s,\eta^s) \right] \right\}.
\end{aligned}$$

The first-order condition with respect to consumption is given by

$$\beta^t u'(c(z^t,\eta^t)) = \zeta(z^t,\eta^t) P_x(z^t) \;\; for \;\; all \;\; (z^t,\eta^t),$$

where $\zeta(z^t, \eta^{h,t})$ is defined as

$$\zeta_t(z^t,\eta^t) = \zeta_{t-1}(z^{t-1},\eta^{t-1}) + \nu_t(z^t,\eta^t) - \varphi_t(z^t,\eta^t).$$

Therefore, the first-order condition with respect to consumption is independent of trading restrictions. The first-order condition with respect to total asset holdings at the end of period $t-1$,

$\widehat{a}_{t-1}(z^{t-1}, \eta^{t-1})$, is

$$\sum_{(z^t,\eta^t)} R^p_{t,t-1}(z^t)\nu_t(z^t,\eta^t)P(z^t)\pi(z^{t,}\eta^t) = 0 \;\; for \;\; all \;\; z^t, \eta^t.$$

This condition varies according to different trading restrictions.

A.2.3 First-Order Condition of Foreign Households

Similarly, the first-order condition (FOC) respect to consumption for all foreign investors is

$$\beta^t u'(c^*(z^t,\eta^t)) = \zeta^*(z^t,\eta^t)P^*(z^t) \;\; for \;\; all \;\; (z^t,\eta^t),$$

and the first-order condition with respect to the asset choice for foreign non-Mertonian traders is

$$\sum_{(z^t,\eta^t)} R^{p,*}_{t,t-1}(z^t)\nu^*_t(z^t,\eta^t)P^*(z^t)\pi(z^{t,}\eta^t) = 0 \;\; \text{for all} \;\; z^t, \eta^t.$$

A.3 Stochastic Discount Factor

By summing the first-order conditions with respect to consumption, equation (16), across all domestic households at period t, we can obtain the price of home consumption basket at state z^t:

$$P(z^t) = \beta^t C(z^t)^{-\gamma} h_t(z^t)^{\gamma}$$

where $h_t(z^t)$ is defined as $h_t(z^t) \equiv \sum_{\eta^t} \zeta(z^t,\eta^t)^{-\frac{1}{\gamma}}$.

Therefore, the home stochastic discount factor is given by the Breeden-Lucas stochastic discount factor (SDF) with a multiplicative adjustment:

$$Q_{t+1}(z^{t+1}|z^t) \equiv \frac{P(z^{t+1})}{P(z^t)} = \beta \left(\frac{C(z^{t+1})}{C(z^t)} \right)^{-\gamma} \left(\frac{h_{t+1}(z^{t+1})}{h_t(z^t)} \right)^{\gamma}. \tag{17}$$

Similarly, we can derive the stochastic discount factor of the foreign country:

$$Q^*_{t+1}(z^{t+1}|z^t) \equiv \frac{P^*(z^{t+1})}{P^*(z^t)} = \beta \left(\frac{C^*(z^{t+1})}{C^*(z^t)} \right)^{-\gamma} \left(\frac{h^*_{t+1}(z^{t+1})}{h^*_t(z^t)} \right)^{\gamma} \tag{18}$$

A.4 Price of the Final Consumption Good

We start by analyzing the home country's price index. Let $P(z^t)$ be the price level in the home country. The price level represents the minimum expenditure on $c(z^t) = c_n(z^t, \eta^t)^{\theta} c_x(z^t, \eta^t)^{1-\theta}$. To find the price level,we solve the cost minimization problem:

$$\max_{Y_n(z^t), Y^*_x(z^t)} P(z^t) Y_n(z^t)^{\theta} Y^*_x(z^t)^{1-\theta} - P_n(z^t) Y_n(z^t) - P_x(z^t) Y^*_x(z^t).$$

The first-order condition implies that

$$\frac{P_n(z^t)}{P(z^t)} = \theta Y_n(z^t)^{\theta-1} Y^*_x(z^t)^{1-\theta} = \theta \frac{C(z^t)}{Y_n(z^t)}$$

$$\frac{P_x(z^t)}{P(z^t)} = (1-\theta) Y_n(z^t)^{\theta} Y^*_x(z^t)^{-\theta} = (1-\theta) \frac{C(z^t)}{Y^*_x(z^t)}$$

Next, we analyze the same problem for the foreign country:

$$\frac{P^*_n(z^t)}{P^*(z^t)} = \theta Y^*_n(z^t)^{\theta-1} Y_x(z^t)^{1-\theta} = \theta \frac{C^*(z^t)}{Y^*_n(z^t)}$$

The first-order condition implies that

$$\frac{P_x^*(z^t)}{P^*(z^t)} = (1-\theta)Y_n^*(z^t)^\theta Y_x(z^t)^{-\theta} = (1-\theta)\frac{C^*(z^t)}{Y_x(z^t)}.$$

B Computational Algorithm

In the spirit of Chien, Cole, and Lustig (2011), we develop the following computational algorithm.

Algorithm 1. *Computational algorithm:*

1. *Guess a function for the exchange rates $e_t(z_k)$ as well as the updating rule at home $\frac{h(z'_k)}{h(z_k)}$ and abroad $\frac{h^*(z'_k)}{h^*(z_k)}$.*

2. *Since the consumption process is governed by the endowment shock process, the state-contingent price is determined by*

$$Q(z'_k; z_k) \equiv \frac{P(z'_k)}{P(z_k)} = \beta \left(\frac{C(z'_k)}{C(z_k)}\right)^{-\gamma} \left(\frac{h(z'_k)}{h(z_k)}\right)^{\gamma}$$

$$Q^*(z'_k; z_k) \equiv \frac{P^*(z'_k)}{P^*(z_k)} = \beta \left(\frac{C^*(z'_k)}{C^*(z_k)}\right)^{-\gamma} \left(\frac{h^*(z'_k)}{h^*(z_k)}\right)^{\gamma}.$$

3. *The level of the exchange rate is given by*

$$e_t(z_k) = \frac{P(z_k)}{P^*(z_k)} = \frac{C(z_k)^{-\gamma} h_t(z_k)^\gamma}{C^*(z_k)^{-\gamma} h_t^*(z_k)^\gamma} = \left(\frac{m_t^*(z_k)}{m_t(z_k)}\right)^{-\gamma(1-2\theta)} \frac{h_t(z_k)^\gamma}{h_t^*(z_k)^\gamma}.$$

4. *Given prices, solve the optimization problem of each individual investor.*

5. *Simulate the economy and derive the implied new exchange rate function, $e_t(z_k)$, and the new*

updating rule for $\frac{h(z'_k)}{h(z_k)}$ *and* $\frac{h^*(z'_k)}{h^*(z_k)}$

6. *Compare the original guess with the new implied value. If they are the same, we have an equilibrium; otherwise, we iterate* $e_t(z_k)$, $\frac{h(z'_k)}{h(z_k)}$, *and* $\frac{h^*(z'_k)}{h^*(z_k)}$.

In our simulation panel, we cannot observe h and h^* in levels; h and h^* are nonstationary, but we need the level of h to compute the exchange rate. However, h and h^* have to share a common stochastic trend, given that e is stationary. We can derive the growth rate of h and h^* as a function of the truncated aggregate histories. The stationarity of e_t implies that it can be stated as a function of the truncated aggregate history:

$$\ln\left(\frac{h(z'_k)}{h(z_k)}\right) - \ln\left(\frac{h^*(z'_k)}{h^*(z_k)}\right) = b(z'_k) - b(z_k) \equiv \Delta(z'_k, z_k). \tag{19}$$

If we know $b(z_k)$ for all possible $z_k \in \mathcal{A}$, then we know the exchange rate $e(z_k)$ and $\Delta(z'_k, z_k)$. We use $\overline{H}(z'_k, z_k)$ to denote the average growth rate:

$$\overline{H}(z'_k, z_k) \equiv \frac{\ln\left(\frac{h(z'_k)}{h(z_k)}\right) + \ln\left(\frac{h^*(z'_k)}{h^*(z_k)}\right)}{2}. \tag{20}$$

From equations (19) and (20), we know that the growth rate of h in a particular country is given by:

$$\begin{aligned}
\ln\left(\frac{h(z'_k)}{h(z_k)}\right) &= \overline{H}(z'_k, z_k) + \frac{\Delta(z'_k, z_k)}{2} \\
\ln\left(\frac{h^*(z'_k)}{h^*(z_k)}\right) &= \overline{H}(z'_k, z_k) - \frac{\Delta(z'_k, z_k)}{2}.
\end{aligned}$$

Given this investment in notation, we can now describe the updating process $\frac{h(z'_k)}{h(z_k)}$, $\frac{h^*(z'_k)}{h^*(z_k)}$, and

$e_t(z^t)$

Algorithm 2. *Simulation steps:*

1. *In each z_k, update the growth rate of h and h^*, denoted by* $\ln\left(\frac{h(z_k')}{h(z_k)}\right)_{sim}$ *and* $\ln\left(\frac{h^*(z_k')}{h^*(z_k)}\right)_{sim}$, *respectively.*

2. *From equation (20), the updated guess of $\overline{H}$ is given by*

$$\overline{H}_{new}(z_k', z_k) = \frac{\ln\left(\frac{h(z_k')}{h(z_k)}\right)_{sim} + \ln\left(\frac{h^*(z_k')}{h^*(z_k)}\right)_{sim}}{2}.$$

3. *The updated guess of $b_{new}(z_k)$ is the solution of the following minimization problem:*

$$b_{new}(z_k) = \arg\min_{b(z_k)} \left(E\left[\left(\frac{h(z_k')}{h(z_k)}\right) - \left(\frac{h(z_k')}{h(z_k)}\right)_{sim}\right]^2 + E\left[\left(\frac{h^*(z_k')}{h^*(z_k)}\right) - \left(\frac{h^*(z_k')}{h^*(z_k)}\right)_{sim}\right]^2 \right),$$

such that

$$\begin{aligned} \ln\left(\frac{h(z_k')}{h(z_k)}\right) &= \overline{H}_{new}(z_k', z_k) + \frac{\Delta(z_k', z_k)}{2} \\ \ln\left(\frac{h^*(z_k')}{h^*(z_k)}\right) &= \overline{H}_{new}(z_k', z_k) - \frac{\Delta(z_k', z_k)}{2} \end{aligned}$$

and

$$\Delta(z_k', z_k) = b(z_k') - b(z_k).$$

4. *The updated guess of $e(z_k)$ and the growth rate of h and h^* are therefore*

$$\ln e_{new}(z_k) = \gamma(2\theta - 1)\ln\left(\frac{m^*(z_k)}{m(z_k)}\right) + \gamma b_{new}(z_k) \tag{21}$$

$$\ln\left(\frac{h(z_k')}{h(z_k)}\right)_{new} = \overline{H}_{new}(z_k', z_k) + \frac{\Delta_{new}(z_k', z_k)}{2}$$
$$\ln\left(\frac{h^*(z_k')}{h^*(z_k)}\right)_{new} = \overline{H}_{new}(z_k', z_k) - \frac{\Delta_{new}(z_k', z_k)}{2},$$

where

$$\Delta_{new}(z_k', z_k) = b_{new}(z_k') - b_{new}(z_k).$$